

# Computation of the winding number diffusion rate due to the cosmological sphaleron

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## Abstract

A detailed quantitative analysis of the transition process mediated by a sphaleron type non-Abelian gauge field configuration in a static Einstein universe is carried out. By examining spectra of the fluctuation operators and applying the zeta function regularization scheme, a closed analytical expression for the transition rate at the one-loop level is derived. This is a unique example of an exact solution for a sphaleron model in  $3 + 1$  spacetime dimensions.

# 1 Introduction

The discovery of vacuum periodicity [1], [2] and sphalerons [3] in the standard model has revealed a considerable fermion number violation in the theory at high temperatures. The fermion number changes in the processes of barrier transitions between the distinct topological sectors of the theory, where the barrier height is determined by the sphaleron energy. Such processes become unsuppressed at temperatures of the order of the sphaleron mass. Since the different topological sectors are in thermal equilibrium at high temperatures, the sphaleron transitions lead to the dissipation of the baryon asymmetry produced at earlier times, that is, during the electroweak phase transition or at GUT energies [4]. The estimates for the transition rate [5]-[9] show that the sphaleron mechanism can be efficient enough to reduce the asymmetry by several orders of magnitude.

The sphaleron solution in the standard model is known only numerically, which causes considerable computational difficulties [7]-[9]. Other field-theoretical sphaleron models have therefore become important [10], [11], [12]. Some of these models are exactly solvable in the sense that the corresponding sphaleron transition rate can be evaluated analytically at the one-loop level [11], [12], which provides a closer insight into the physics. Unfortunately, all of these models exist only in low spacetime dimensions. There is also an alternative example of four dimensional sphalerons which appear in the theory of a self-gravitating non-Abelian gauge field. Gravity violates the scale invariance and plays therefore a role similar to that of a Higgs field. This results in the existence of classical [13] sphaleron-like solutions in the theory [14], [15], which are also known only numerically.

The purpose of the present paper is to investigate a sphaleron model that is exactly solvable and yet exists in  $3 + 1$  spacetime dimensions, which distinguishes it from the other known models. Similarly to the example mentioned above, this model deals with a non-Abelian gauge field interacting with gravity, but now gravity is regarded as a fixed external field. Specifically, we shall consider the theory of a pure  $SU(2)$  Yang-Mills field in a static Einstein universe. This theory admits an analytically known sphaleron solution which we shall call cosmological sphaleron. The analysis of the corresponding transition problem is the subject of this paper.

In fact, this type of sphaleron was found long ago. The solution itself has been discussed in various contexts, together with the other related solutions of the (Einstein)-Yang-Mills field equations [16]-[18]. The sphaleron nature of the solution was also recognized and discussed in connection with the finite volume QCD [19] and in a cosmological context as well [20], [21]. However, an analysis of the corresponding sphaleron transition problem has been lacking so far.

The cosmological sphaleron is distinguished by the important property that the sphaleron configuration consists of the gauge field alone. This implies that the sphaleron is not plagued with the various symmetry breaking phase transitions, and this also ensures that the dynamics of the sphaleron mediated processes is conformally invariant – up to the anomalous scale dependence of the gauge coupling constant. Since the sphaleron has very high symmetry, there is an opportunity to analyze the sphaleron transition problem analytically at the one-loop level, which is of great methodological interest. (We shall use the names “cosmological” and “universe”, although the model can be applied not only in the cosmological context.)

The problem which will be investigated below can be formulated as follows. Consider a gauge field in a static Einstein universe at finite temperature. Specifically, consider the thermal ensemble over one of the topological vacua of the field. Find the rate of the decay of this thermal state due to the diffusion into the neighboring topological sector. In order to obtain the answer, we use the Langer-Affleck formula and take only the bosonic degrees of freedom into account (the fermion contribution can be considered in a similar way). We do not assume the high temperature limit and take the sum over all Matsubara modes. We use the zeta function regularization scheme, which allows us to entirely carry out the analysis. Our principal results are given by Eqs.(6.20)-(6.27) and Eq.(6.37) and presented in Figs.1-3. The rest of the paper is organized as follows. The basic properties of the model, such as topological vacua and the sphaleron solution are discussed in Sec.II. A brief derivation, based on the path integral methods, of the decay rate of an unstable phase at finite temperature is given in Sec.III. The path integration procedure is outlined in Sec.IV. The spectra of the fluctuation operators are analyzed in Sec.V. The determinants of these operators are calculated in Sec.VI which also includes the consideration of the high temperature limit. Sec.VII. contains some concluding remarks, and the derivation of numerous formulae used in the zeta function approach is given in the Appendix.

Throughout the paper the units  $\hbar = c = \kappa_B = 1$  are used. The symbol  $g$  stands for the gauge coupling constant, whereas the spacetime metric is denoted by  $\mathbf{g}$ .

## 2 The sphaleron on $S^3$

Consider the static Einstein universe  $(M, \mathbf{g})$ , where  $M = R^1 \times S^3$ , and the metric is

$$ds^2 = \mathbf{a}^2(-d\eta^2 + d\Omega_3^2). \quad (2.1)$$

Here  $\mathbf{a}$  is a constant scale factor, and the line element on  $S^3$  is parameterized by

$$d\Omega_3^2 = d\xi^2 + \sin^2\xi(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (2.2)$$

where  $\xi \in [0, \pi]$ , and  $\vartheta, \varphi$  are the usual spherical coordinates on  $S^2$ .

The model under consideration is defined by the action

$$S[A] = -\frac{1}{2g^2} \text{tr} \int_M F_{\mu\nu} F^{\mu\nu} \sqrt{-\mathbf{g}} d^4x, \quad (2.3)$$

where  $A = A_\mu dx^\mu = T_p A_\mu^p dx^\mu$  is the gauge field, and  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$  is the field tensor. The hermitian group generators are  $T_p = \tau^p/2$  with  $\tau^a$  being the Pauli matrices. The metric  $\mathbf{g}$  in Eq.(2.3) is given by Eq.(2.1), such that the action defines the theory of a pure non-Abelian  $SU(2)$  gauge field in the static Einstein universe. (Throughout the paper we neglect the back reaction of the gauge field on the spacetime geometry, which can be justified if  $\mathbf{a} \gg l_{pl}/g$ , where  $l_{pl}$  is Planck's length.) The classical equations of motion following from the action are

$$\nabla_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0, \quad (2.4)$$

where  $\nabla_\mu$  is the covariant derivative with respect to the spacetime metric.

First, we need to describe the topological vacua of the gauge field in this case. We do so by introducing a smooth, time-independent function on the manifold,  $U(\mathbf{x}) \in SU(2)$ , where  $\mathbf{x} = x^m$ ,  $m = 1, 2, 3$ , thus defining the mapping

$$U(\mathbf{x}) : S^3 \rightarrow SU(2). \quad (2.5)$$

Any such mapping can be characterized by an integer winding number

$$\mathbf{k}[U] = \frac{1}{24\pi^2} \text{tr} \int_{S^3} U dU^{-1} \wedge U dU^{-1} \wedge U dU^{-1}, \quad (2.6)$$

such that the set of all  $U$ 's falls into a countable sequence of disjoint homotopy classes. The representative of the  $k$ -th class,  $U^{(k)}$ ,  $\mathbf{k}[U^{(k)}] = k$ , can be chosen as

$$U^{(k)}(\mathbf{x}) = U^{(k)}(\xi, \vartheta, \varphi) = \exp \{ -ik\xi n^a \tau^a \}, \quad (2.7)$$

where  $n^a = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ . The functions  $U$  generate static gauge transformations,

$$A \rightarrow UAU^{-1} + iUdU^{-1}. \quad (2.8)$$

The elements of the zeroth homotopy class,  $U^{(0)}$ , give rise to small gauge transformations which can be continuously deformed to identity; other functions,  $U^{(k)}$ ,  $k \neq 0$ , generate large transformations. A vacuum of the gauge field is a pure gauge,  $A_{vac} = iUdU^{-1}$ . Since all  $U$ 's split into homotopy classes, all pure gauges decompose into disjoint sets called topological vacua. The  $k$ -th topological vacuum in the temporal gauge is

$$A^{(k)}(\mathbf{x}) = iU^{(k)}dU^{(k)-1}. \quad (2.9)$$

By construction, the Chern-Simons number of this field configuration coincides with the winding number  $k$ .

Distinct topological vacua can not be joined by a continuous interpolating sequence of pure gauge configurations,  $iUdU^{-1}$ , since this would require a change in the winding number  $\mathbf{k}[U]$ . However, one can join them by a family of non-vacuum fields. That is how one can see that the model admits a sphaleron solution. Consider the two neighboring vacua given by Eqs.(2.7), (2.9):  $A^{(0)} = 0$  and  $A^{(1)}$ . They can be joined by the following path in the configuration space:

$$A[h] = i\frac{1+h}{2}U^{(1)}dU^{(1)-1}, \quad (2.10)$$

where the parameter  $h \in [-1, 1]$ . (Applying a large gauge transformation one can reduce such a path to a non contractible loop.) The energy-momentum tensor for this field is

$$T^\mu_\nu(A[h]) = \frac{(h^2 - 1)^2}{2g^2 \mathbf{a}^4} \text{diag}(3, -1, -1, -1), \quad (2.11)$$

such that the energy is given by

$$E[h] = \int T^0_0 \sqrt{3} \mathbf{g} d^3x = \frac{3\pi^2}{g^2 \mathbf{a}} (h^2 - 1)^2. \quad (2.12)$$

This function has the typical barrier shape: it vanishes at the vacuum values of  $h$ ,  $h = \pm 1$ , and reaches its maximum in between, at  $h = 0$ . The top of the barrier relates to the field configuration

$$A^{(sp)} \equiv A[h = 0] = \frac{i}{2} U^{(1)}dU^{(1)-1}, \quad (2.13)$$

with the energy  $E_{max} = 3\pi^2/g^2\mathbf{a}$ .

Similarly, one can define  $E_{max}$  for any other interpolating path, and then minimize the result over all paths. If a non zero minimum exists then it relates to an unstable classical solution called sphaleron. By construction, the sphaleron energy defines the minimal height of the potential barrier [3], [10].

To carry out such a program would, however, be too difficult a task which has never been done in reality. Instead, we simply check that the field (2.13) solves the classical equations of motion. To see this, it is illuminating to allow the parameter  $h$  in Eq.(2.10) to depend on time,  $h \rightarrow h(\eta)$ . Then the Yang-Mills equations (2.4) for the field (2.10) reduce to one non-trivial equation, which admits a first integral:

$$\frac{d^2h}{d\eta^2} + 2h(h^2 - 1) = 0, \quad \Rightarrow \quad \left(\frac{dh}{d\eta}\right)^2 + (h^2 - 1)^2 = \varepsilon, \quad (2.14)$$

with  $\varepsilon$  being an integration constant. Effectively, these equations describe a particle moving in the one-dimensional double-well potential. When  $\varepsilon = 1$ , one finds the static solution  $h(\eta) = 0$ , which describes an unstable equilibrium of the particle on the top of the barrier. This shows that field configuration (2.13) indeed solves the equations of motion and relates to a saddle point of the energy functional, such that it can be naturally called sphaleron. Later we shall see that this solution has only one unstable mode. In addition, for  $\varepsilon = 1$ , Eqs.(2.14) admit the solution for the particle rolling down the barrier,  $h(\eta) = \sqrt{2}/\cosh \sqrt{2}(\eta - \eta_0)$ , which describes the time evolution of the sphaleron during its classical decay.

Of course, these arguments do not prove that the sphaleron relates to the absolute minimum of energy for static, nonvacuum solutions. Notice however, that (2.13) is the only static, nonvacuum,  $SO(4)$  symmetric solution [17]. It is therefore very plausible that this solution does indeed minimize the energy.

Another handy form for the sphaleron solution (2.13) can be achieved as follows. Introduce the left and right invariant 1-forms on  $S^3$ ,

$$\omega_L^a = \frac{i}{2}tr(\tau^a U^{(1)} dU^{(1)-1}), \quad \omega_R^a = \frac{i}{2}tr(\tau^a U^{(1)-1} dU^{(1)}), \quad (2.15)$$

which satisfy the Maurer-Cartan equations

$$d\omega^a + \varepsilon_{abc} \omega^b \wedge \omega^c = 0. \quad (2.16)$$

This allows us to represent the field (2.13) as

$$A^{(sp)} = T_a \omega_L^a. \quad (2.17)$$

It is worth noting the following feature of this solution: the sphaleron configuration consists of the gauge field alone. This, together with the high symmetry of the solution, will be of crucial importance for the analysis below. It is well known that in the Minkowski space, the existence of static, finite energy solutions for the pure gauge field is ruled out by the scaling arguments. However, these arguments do not generally apply in curved spacetime, since the invariance with respect to the rescaling of the coordinates,  $\mathbf{x} \rightarrow \lambda\mathbf{x}$ , is broken by the curvature. It is therefore gravity which ensures the existence of the static sphaleron solution. Since the spacetime geometry is homogeneous

and isotropic, the sphaleron inherits the same symmetries, such that, for instance, its energy-momentum tensor has the manifest  $SO(4)$ -symmetric structure.

Let us mention also the following point: since the Yang-Mills equations are conformally invariant, and the geometry (2.1) is conformally flat, there exists the Minkowski space counterpart of the sphaleron solution. However, this flat spacetime solution is, of course, no longer static. Specifically, if one chooses the conformal factor of the metric (2.1) as  $\mathbf{a} = \mathbf{a}(\eta, \xi) = (\cos \eta + \cos \xi)^{-1}$  and introduces the new coordinates  $t \pm r = \tan((\eta \pm \xi)/2)$ , the metric assumes the standard flat form [20]. In the new coordinates, the sphaleron field (2.13) becomes a member of the family of the elliptic solutions [22], which describe spherical shells of the Yang-Mills radiation in the Minkowski space [23]. In this sense, one can think of the sphaleron as being a radiative solution which is rendered static by gravity.

We will also need some knowledge about instantons in the model. We first note that the vacua and the sphaleron, since they are static, can be regarded as Euclidean solutions. Next, let us pass to the imaginary time in Eq.(2.14),  $\eta \rightarrow -i\tau$ ,

$$\left(\frac{dh}{d\tau}\right)^2 - (h^2 - 1)^2 = -\varepsilon. \quad (2.18)$$

Apart from the static solutions, this equation admits also the interpolating solutions for  $\varepsilon = 0$ , such that  $h(-\infty) = -1$ ,  $h(\infty) = 1$ . The periodic solutions exist for  $0 < \varepsilon < 1$ , and the corresponding period is bounded from below,

$$\tau > \sqrt{2}\pi. \quad (2.19)$$

Other known instanton solutions on  $S^3$  can be obtained from the flat space BPST instantons by making use of the conformal invariance of the YM equations [19].

### 3 The sphaleron transition rate

Consider the low energy excitations over the  $k$ -th topological vacuum,  $A^{(k)}(\mathbf{x}) \rightarrow A^{(k)}(\mathbf{x}, t) = A^{(k)}(\mathbf{x}) + \delta A(\mathbf{x}, t)$ . If the energy is small compared to the barrier height,  $3\pi^2/g^2\mathbf{a}$ , then the excitations over the distinct vacua are classically independent. There is a nonzero amplitude for the quantum tunneling between distinct sectors, however the corresponding probability is exponentially small. On the perturbative level, one can consider the excitations in each sector independently. The energy of the ground state excitation in each sector is  $1/\mathbf{a}$  (see Eq.(5.20) below). The necessary condition for the smallness of the energy of the excitations is therefore  $g^2/3\pi^2 \ll 1$ .

Consider the zeroth topological sector and assume a thermal distribution for the states in this sector. There is a finite probability for such a thermal system to decay, both because of the underbarrier tunneling and due to the overbarrier thermal excitation. According to the Langer-Affleck theory of the metastable phase [28], the decay rate is proportional to the imaginary part of the free energy. To estimate the latter, it is convenient to use the path integral approach (the precise definition of the path integration procedure will be given in the next section).

The partition function of the gauge field is

$$Z = \exp(-\beta F) = \int d[A] \exp(-S_E[A]). \quad (3.1)$$

In this expression,  $S_E[A]$  is the Euclidean action of the gauge field in the static Riemannian space  $(S^1 \times S^3, \mathbf{g})$ , with  $\mathbf{g}$  being the analytic continuation of the metric (2.1) to the imaginary time,  $\eta \rightarrow -i\tau$ ,  $\tau \in [0, \beta]$ . In the weak coupling limit, one can approximate the partition function by the sum over the classical extrema,  $A^{\{j\}}$ , as  $Z \simeq \sum_j Z_j$ , where the semiclassical contribution of the  $j$ -th extremum is

$$Z_j = \exp(-\beta F_j) = \exp(-S[A^{\{j\}}]) \int d[\varphi] \exp(-\delta^2 S_j). \quad (3.2)$$

The action for the fluctuations around the  $j$ -th extremum,  $A^{\{j\}} \rightarrow A^{\{j\}} + \varphi$ , can be represented as

$$S[A^{\{j\}} + \varphi] = S[A^{\{j\}}] + \delta^2 S_j + \dots, \quad \delta^2 S_j = \int_0^\beta d\tau \int_{S^3} (\varphi, \hat{\mathbf{D}}_j \varphi) \sqrt{\mathbf{g}} d^3x, \quad (3.3)$$

where  $\hat{\mathbf{D}}_j = \hat{\mathbf{D}}[A^{\{j\}}]$  is the Gaussian fluctuation operator. This gives the one loop expression for the partition function,

$$Z \simeq \sum_j Z_j = \sum_j \frac{\exp(-S[A^{\{j\}}])}{\sqrt{\text{Det}(\hat{\mathbf{D}}_j)}}. \quad (3.4)$$

Assume that this sum is dominated by two terms,  $Z \simeq Z_0 + Z_1$ , where  $Z_0$  and  $Z_1$  are the contributions of the vacuum and the sphaleron, respectively. Other periodic instantons that could exist for a given value of  $\beta$  are assumed to have a large action.

The sphaleron fluctuation operator  $\hat{\mathbf{D}}_1$  has at least one negative eigenvalue  $\omega_-^2 < 0$ . Under the condition specified by the lower bound in Eq.(3.8) below, there is only one negative eigenvalue. This implies that  $Z_1$  is purely imaginary, and the free energy of the whole system picks up the imaginary part

$$\text{Im}(\beta F) = -\text{Im} \ln Z \simeq -\text{Im} \ln \left( Z_0 \left( 1 + \frac{Z_1}{Z_0} \right) \right) \simeq -\frac{1}{Z_0} \text{Im} Z_1. \quad (3.5)$$

According to Langer [28], the imaginary part of the free energy is to be interpreted as giving rise to the decay rate of the unstable phase built over the perturbative vacuum as

$$\Gamma = \frac{|\kappa|}{\pi T} \text{Im} F, \quad (3.6)$$

where the damping constant  $\kappa$  is the real time decay rate of the sphaleron configuration in the heat bath. In the weak coupling limit one has  $|\kappa| = |\omega_-|$  [5], which finally determines the decay rate to be

$$\Gamma = -\frac{|\omega_-|}{\pi} \frac{\text{Im} Z_1}{Z_0}. \quad (3.7)$$

This formula holds in the following range of temperatures [28]:

$$\frac{|\omega_-|}{2\pi} < \frac{1}{\beta} \ll \frac{3\pi^2}{g^2}. \quad (3.8)$$

The lower bound rules out the periodic instantons which play the leading role at low temperatures [28] (for the cosmological sphaleron one has  $|\omega_-| = \sqrt{2}$ , such that this condition is opposite to that specified by (2.19)). The upper bound is the sphaleron energy which must exceed the temperature, otherwise the system would not be metastable. Notice that this formula involves only conformally invariant quantities, the conformal factor

$\mathbf{a}$  drops out. Another crucial assumption is the weak coupling limit,  $g^2/4\pi \ll 1$ . First of all, this ensures the very existence of the thermal ensemble. In addition, it justifies the validity of the Gaussian approximation, and, moreover, leads to the weak damping limit for  $\kappa$ .

## 4 The path integration procedure

We outline below the main steps of the path integration procedure. It is worth noting that the gauge field theory on  $S^3$  resembles that on  $S^4$ . We shall therefore mainly follow the approach given in Refs.[25]–[27].

Passing to the imaginary time, the spacetime metric (2.1) becomes

$$ds^2 = \mathbf{a}^2(d\tau^2 + d\Omega_3^2), \quad (4.1)$$

where  $\tau \in [0, \beta]$ . We assume that the coordinates are dimensionless, implying that  $[\mathbf{a}] = [L]$ . Notice that  $1/\beta$  is the *conformal* temperature. The physical temperature  $T$  is defined with respect to the physical time,  $\mathbf{a}\tau$ , such that  $T = 1/\beta\mathbf{a}$ .

The Euclidean action of the gauge field is

$$S_E[A] = \frac{1}{2g^2} \text{tr} \int F_{\mu\nu} F^{\mu\nu} \sqrt{\mathbf{g}} d^4x > 0. \quad (4.2)$$

Consider small fluctuations around the  $j$ -th extremum of the action  $A_\mu^{\{j\}} \rightarrow A_\mu^{\{j\}} + \varphi_\mu$ , where the values  $j = 1, 0$  refer to the sphaleron and the vacuum configurations, respectively (we shall omit this index where possible). The infinitesimal gauge transformations act as  $\varphi_\mu \rightarrow \varphi_\mu^{(\alpha)} = \varphi_\mu + D_\mu \alpha$ , where  $D_\mu \alpha = \nabla_\mu \alpha - i[A_\mu, \alpha]$ , and  $\alpha$  is a Lie algebra valued scalar field. Define the following operators:

$$\hat{\mathbf{D}}\varphi^\nu = \hat{\mathbf{M}}\varphi^\nu + D^\nu(D_\sigma\varphi^\sigma), \quad \hat{\mathbf{M}}\varphi^\nu = -D_\sigma D^\sigma\varphi^\nu + R_\sigma^\nu\varphi^\sigma + 2i[F_\sigma^\nu, \varphi^\sigma]. \quad (4.3)$$

These are the vector fluctuation operator and the gauge fixed fluctuation operator, respectively. Introduce also the Faddeev-Popov operator

$$\hat{\mathbf{M}}^{FP}\alpha = -D_\sigma D^\sigma\alpha. \quad (4.4)$$

In these formulas  $R_\sigma^\nu$  is the Ricci tensor for the geometry (4.1), and  $F_{\nu\sigma}$  is the background gauge field tensor. These operators are self-adjoint (symmetric) with respect to the scalar products

$$\langle\varphi, \varphi'\rangle = 2 \text{tr} \int \varphi^\sigma \varphi'_\sigma \sqrt{\mathbf{g}} d^4x, \quad \langle\alpha, \alpha'\rangle = 2 \text{tr} \int \alpha \alpha' \sqrt{\mathbf{g}} d^4x. \quad (4.5)$$

The norms are  $\|\varphi\| = \sqrt{\langle\varphi, \varphi\rangle}$ , and  $\|\alpha\| = \sqrt{\langle\alpha, \alpha\rangle}$ . The action can be expanded as

$$S_E[A + \varphi] \approx S_E[A] + \frac{1}{2g^2} \langle\varphi, \hat{\mathbf{D}}\varphi\rangle. \quad (4.6)$$

The Gaussian path integral is

$$Z = \exp(-S_E[A]) \int d^{FP}[\varphi] \exp\left(-\frac{1}{2g^2} \langle\varphi, \hat{\mathbf{D}}\varphi\rangle\right), \quad (4.7)$$



where the Faddeev-Popov measure is

$$d^{FP}[\varphi] = d[\varphi] \mathcal{G}(\varphi) \mathcal{F}(\varphi), \quad 1 = \mathcal{G}(\varphi) \int d[\alpha] \mathcal{F}(\varphi^{(\alpha)}), \quad (4.8)$$

and  $\mathcal{F}$  is the gauge fixing function. The fluctuations  $\varphi$  can be decomposed as

$$\varphi_\mu = D_\mu \alpha + \xi_\mu, \quad (4.9)$$

where the pure gauge part  $D_\mu \alpha$  is annihilated by  $\hat{\mathbf{D}}$ , and  $\xi_\mu$  is orthogonal to all gauge modes,  $D_\sigma \xi^\sigma = 0$ . The fields  $\xi_\mu$  and  $\alpha$  can be expanded with respect to the eigenfunctions of  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{M}}^{FP}$ :

$$\xi^\mu = \sum_k C_k \xi_k^\mu, \quad \alpha = \sum_n B_n \alpha_n; \quad \hat{\mathbf{D}} \xi_k^\nu = \lambda_k \xi_k^\nu, \quad \hat{\mathbf{M}}^{FP} \alpha_n = q_n \alpha_n. \quad (4.10)$$

The gauge fixing function is chosen to be

$$\mathcal{F}(\varphi) = \exp \left\{ -\frac{1}{2g^2} \langle D_\sigma \varphi^\sigma, D_\mu \varphi^\mu \rangle \right\} = \exp \left\{ -\frac{1}{2g^2} \sum_n B_n^2 q_n^2 |\alpha_n|^2 \right\}, \quad (4.11)$$

and the integration measure is the square root of the determinant of the metric on the function space [27]:

$$d[\varphi] = \prod_k \frac{\mu_0}{\sqrt{2\pi g}} dC_k |\xi_k^\mu| \prod_n' \frac{\mu_0}{\sqrt{2\pi g}} dB_n \sqrt{q_n} |\alpha_n|, \quad d[\alpha] = \prod_n \frac{\mu_0^2}{\sqrt{2\pi g}} dB_n |\alpha_n|. \quad (4.12)$$

Here  $\mu_0/(\sqrt{2\pi g})$  is a normalization factor with  $\mu_0$  being an arbitrary normalization scale; the prime indicates that terms with  $q_n = 0$  should be omitted. Taking (4.8)-(4.12) into account, the Gaussian path integral in (4.7) reduces to

$$\mathcal{G} \int \prod_k \frac{\mu_0 dC_k}{\sqrt{2\pi g}} |\xi_k^\mu| \prod_n' \frac{\mu_0 dB_n}{\sqrt{2\pi g}} |\alpha_n| \exp \left\{ -\frac{1}{2g^2} \left( \sum_k \lambda_k C_k^2 |\xi_k^\mu|^2 + \sum_n q_n B_n^2 |\alpha_n|^2 \right) \right\}, \quad (4.13)$$

where  $\sqrt{q_n}$  has been absorbed in  $B_n$ , and

$$\mathcal{G}^{-1} = \int \prod_n \frac{\mu_0^2}{\sqrt{2\pi g}} dB_n |\alpha_n| \exp \left\{ -\frac{1}{2g^2} \sum_n B_n^2 q_n^2 |\alpha_n|^2 \right\}. \quad (4.14)$$

Let us first apply these formulae to the sphaleron. In this case, as we will see in the next section, the vector fluctuation operator has one negative eigenvalue,  $\lambda_- < 0$ , whereas all the other eigenvalues  $\lambda_k, q_n$  in (4.13), (4.14) are positive (zero modes are absent). The integral over  $C_-$  in (4.13) can be defined by analytic continuation [24], and the result is  $\mu_0/2i\sqrt{|\lambda_-|}$ . The rest are well-defined Gaussian integrals. It is convenient to introduce the conformally invariant dimensionless operators  $\hat{M}$  and  $\hat{M}^{FP}$  whose eigenvalues are  $\omega_k^2$  and  $\omega_n^2$ :

$$\hat{\mathbf{M}} = \frac{1}{\mathbf{a}^2} \hat{M}, \quad \hat{\mathbf{M}}^{FP} = \frac{1}{\mathbf{a}^2} \hat{M}^{FP}; \quad \lambda_k = \frac{\omega_k^2}{\mathbf{a}^2}, \quad q_n = \frac{\omega_n^2}{\mathbf{a}^2}. \quad (4.15)$$

This implies that the partition function for the Gaussian fluctuations around the sphaleron is

$$Z_1 = \exp(-S_E[A^{(sp)}]) \frac{\mu_0 \mathbf{a}}{2i\sqrt{|\omega_-|}} \frac{Det(\hat{M}_1^{FP}/\mu_0^2 \mathbf{a}^2)}{\sqrt{Det'(\hat{M}_1/\mu_0^2 \mathbf{a}^2)}}, \quad (4.16)$$

where  $Det'$  has all nonpositive eigenvalues omitted, and the index 1 referring to the sphaleron is restored. Notice that  $Det'(\hat{M})$  must be computed on the space of all vector fluctuations  $\varphi^\nu$ , and not only for those satisfying  $D_\sigma \varphi^\sigma = 0$ .

Consider now the vacuum case. As we will see below, the ghost operator  $\hat{M}^{FP}$  in this case has three zero modes  $\alpha_p = \tau^p/2$  ( $p = 1, 2, 3$ ) related to the global gauge rotations of the vacuum  $A = 0$ . The norm of these modes is

$$||\alpha_p||^2 = \int_0^\beta d\tau \int_{S^3} \sqrt{g} d^3x = 2\pi^2 \beta \mathbf{a}^4. \quad (4.17)$$

The quantity  $\mathcal{G}$  specified by (4.14) then reads

$$\mathcal{G} = \frac{1}{\Upsilon} \prod_n' \frac{q_n}{\mu_0^2} = \frac{1}{\Upsilon} Det' \left( \frac{\hat{\mathbf{M}}^{FP}}{\mu_0^2} \right), \quad (4.18)$$

with

$$\Upsilon = \prod_{p=1}^3 \int \frac{\mu_0^2}{\sqrt{2\pi g}} dB_p ||\alpha_p|| = V_{SU(2)} \frac{\pi \sqrt{\pi}}{g^3} \mu_0^6 \mathbf{a}^6 \beta^{3/2}, \quad (4.19)$$

where the integration over  $B_p$  gives the volume of the stability group, in our normalization it is  $V_{SU(2)} = 16\pi^2$ . In addition, there are three constant vector modes annihilated by the vector fluctuation operator:  $\xi_p^\mu = \delta_0^\mu \tau^p/2$ ,  $||\xi_p^\mu||^2 = 2\pi^2 \beta \mathbf{a}^2$ . Under the gauge transformation generated by

$$U_p(\tau) = \exp \left( i\tau \frac{2\pi}{\beta} l \tau^p \right), \quad U_p(0) = U_p(\beta), \quad l \in \mathcal{Z}, \quad (4.20)$$

these modes change according to

$$U_p(\tau) : \xi_p^\mu = \delta_0^\mu \frac{\tau^p}{2} \rightarrow \delta_0^\mu \frac{\tau^p}{2} \left( 1 + \frac{4\pi l}{\beta} \right). \quad (4.21)$$

This shows that the range of integration over  $dC_p$  in (4.13) is finite,  $C_p \in [0, 4\pi/\beta]$ . The contribution of these three modes to (4.13) therefore is

$$\int \prod_{p=1}^3 \frac{\mu_0 dC_p}{\sqrt{2\pi g}} ||\xi_p^\mu|| = \left( \frac{4\pi}{\beta} \right)^3 \frac{\pi \sqrt{\pi}}{g^3} \mu_0^3 \mathbf{a}^3 \beta^{3/2}. \quad (4.22)$$

All other eigenvalues are positive, which finally gives for the fluctuations around the vacuum

$$Z_0 = \frac{4\pi}{\mu_0^3 \mathbf{a}^3 \beta^3} \frac{Det'(\hat{M}_0^{FP}/\mu_0^2 \mathbf{a}^2)}{\sqrt{Det'(\hat{M}_0/\mu_0^2 \mathbf{a}^2)}}. \quad (4.23)$$

We shall omit below the factor  $\mu_0$ ,  $\mu_0 \mathbf{a} \rightarrow \mathbf{a}$ , such that  $\mathbf{a}$  will be understood as the radius of the universe expressed in units of an *arbitrary* length scale.

## 5 Spectra of the fluctuation operators

To analyze the spectra of the conformally invariant operators  $\hat{M}$  and  $\hat{M}^{FP}$  defined by Eqs.(4.3), (4.4), (4.15), one can put  $\mathbf{a} = 1$  in the line element. We introduce the 1-form

basis  $\{\omega^0, \omega^a\}$  on the spacetime manifold, where  $\omega^0 = d\tau$ , and  $\omega^a = \omega_L^a$  are the left invariant 1-forms given by Eq.(2.15). The metric is

$$ds^2 = \omega^0 \otimes \omega^0 + \omega^a \otimes \omega^a. \quad (5.1)$$

Let  $\{e_0, e_a\}$  be the corresponding dual tetrad; here  $e_a = e_a^L$  are the left invariant vector fields on  $S^3$ . Introduce the right invariant fields  $e_a^R$  dual to the 1-forms  $\omega_R^a$ . The following commutation relations hold

$$[e_a^L, e_b^L] = 2\varepsilon_{abc}e_c^L, \quad [e_a^R, e_b^R] = 2\varepsilon_{abc}e_c^R, \quad [e_a^L, e_b^R] = 0. \quad (5.2)$$

Let  $\nabla_0$  and  $\nabla_a$  denote the covariant derivatives along the tetrad vectors  $\{e_0, e_a\}$ . The following tetrad rotation coefficients do not vanish:

$$\nabla_a e_b = \varepsilon_{abc}e_c, \quad \nabla_a \omega^b = \varepsilon_{abc}\omega_c. \quad (5.3)$$

Let us represent the gauge field of the vacuum and the sphaleron as

$$A^{\{j\}} = j \, \omega^s T_s, \quad (5.4)$$

where  $j = 0, 1$ , respectively. All this suggests to expand the fluctuations as [16]

$$\varphi = \phi^0 \omega^0 + \phi^a \omega^a, \quad (5.5)$$

where  $\phi^0$  and  $\phi^a$  are the scalar and the vector fluctuations, respectively.

Using Eqs.(4.3), (4.4), (5.1)–(5.5), a straightforward calculation shows that the fluctuation operators  $\hat{M}_j$  ( $j = 0, 1$ ) decompose into the direct sum of the two operators acting on the scalar and the vector fluctuations, respectively:

$$\hat{M}_j \phi = \left( \hat{M}_j^{scal} \phi^0 \right) \omega^0 + \left( \hat{M}_j^{vec} \phi^a \right) \omega^a. \quad (5.6)$$

Here the scalar fluctuation operator  $\hat{M}^{scal}$  formally coincides with the ghost operator  $\hat{M}^{FP}$  introduced in the previous section,

$$\hat{M}_j^{scal} \phi^0 = -(\nabla_0 \nabla_0 + \nabla_a \nabla_a) \phi^0 - j \left\{ 2\nabla_s [T_s, \phi^0] + [T_s, [T_s, \phi^0]] \right\}. \quad (5.7)$$

The vector operator reads

$$\begin{aligned} \hat{M}_j^{vec} \phi^a = & -(\nabla_0 \nabla_0 + \nabla_c \nabla_c - 4) \phi^a - 2\varepsilon_{abc} \nabla_b \phi^c + \\ & + j \left\{ -2\nabla_s [T_s, \phi^a] + 4\varepsilon_{abs} [T_s, \phi^b] - [T_s, [T_s, \phi^a]] \right\}. \end{aligned} \quad (5.8)$$

Each of these operators decomposes into the direct sum of a temporal and a spatial part,

$$\hat{M} = -\frac{\partial^2}{\partial \tau^2} + \hat{\mathcal{M}}, \quad (5.9)$$

such that the problem reduces to the study of the corresponding spatial operators  $\hat{\mathcal{M}}_j^{scal}$  and  $\hat{\mathcal{M}}_j^{vec}$ .

We now introduce

$$\mathbf{L}_a = \frac{i}{2} e_a^L, \quad \tilde{\mathbf{L}}_a = \frac{i}{2} e_a^R, \quad \tilde{\mathbf{L}}^2 = \mathbf{L}_a \mathbf{L}_a = \tilde{\mathbf{L}}_a \tilde{\mathbf{L}}_a, \quad (5.10)$$

which are the  $SO(4)$  angular momentum operators, since

$$[\mathbf{L}_a, \mathbf{L}_b] = i\varepsilon_{abc}\mathbf{L}_c, \quad [\tilde{\mathbf{L}}_a, \tilde{\mathbf{L}}_b] = i\varepsilon_{abc}\tilde{\mathbf{L}}_c, \quad [\mathbf{L}_a, \tilde{\mathbf{L}}_b] = 0. \quad (5.11)$$

The commuting operators are  $\vec{\mathbf{L}}^2$ ,  $\mathbf{L}_3$  and  $\tilde{\mathbf{L}}_3$ . The eigenvalues are similar to those for the  $SO(3)$  case, but the angular momentum can now assume both integer and half-integer values:

$$\vec{\mathbf{L}}^2 = l(l+1); \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots; \quad \mathbf{L}_3 = m, \quad \tilde{\mathbf{L}}_3 = \tilde{m}; \quad m, \tilde{m} = -l, -l+1, \dots, l. \quad (5.12)$$

Next, expanding the fluctuations over the basis of the Lie algebra,  $\phi^0 = \phi_p^0 T_p$ ,  $\phi^a = \phi_p^a T_p$ , we define the spin and isospin operators  $\vec{\mathbf{S}}$  and  $\vec{\mathbf{T}}$  by

$$\mathbf{S}_a \phi_p^b = \frac{1}{i} \varepsilon_{abc} \phi_p^c, \quad \mathbf{T}_p \phi_r^a = \frac{1}{i} \varepsilon_{prs} \phi_s^a, \quad \mathbf{T}_p \phi_r^0 = \frac{1}{i} \varepsilon_{prs} \phi_s^0, \quad (5.13)$$

which satisfy the usual commutation relations. One has  $\vec{\mathbf{S}}^2 = \vec{\mathbf{T}}^2 = 2$ , which corresponds to the unit spin and unit isospin, respectively.

Using Eqs.(5.7)-(5.13) one can represent the spatial operators for the fluctuations around the sphaleron ( $j = 1$ ) and the vacuum ( $j = 0$ ) configurations as

$$\begin{aligned} \hat{\mathcal{M}}_1^{vec} &= 2 \left( \vec{\mathbf{L}}^2 + (\vec{\mathbf{L}} + \vec{\mathbf{S}} + \vec{\mathbf{T}})^2 - 1 \right), & \hat{\mathcal{M}}_1^{scal} &= 2 \left( \vec{\mathbf{L}}^2 + (\vec{\mathbf{L}} + \vec{\mathbf{T}})^2 - 1 \right), \\ \hat{\mathcal{M}}_0^{vec} &= 2 \left( \vec{\mathbf{L}}^2 + (\vec{\mathbf{L}} + \vec{\mathbf{S}})^2 \right), & \hat{\mathcal{M}}_0^{scal} &= 4\vec{\mathbf{L}}^2. \end{aligned} \quad (5.14)$$

The fluctuation operators therefore reduce to the combinations of the angular momentum operators whose spectra can be analyzed by the usual methods. Let us illustrate the procedure for the operator  $\hat{\mathcal{M}}_1^{vec}$ .

We represent the operator in the form  $\hat{\mathcal{M}}_1^{vec} = 2 \left( \vec{\mathbf{L}}^2 + \vec{\mathbf{J}}^2 - 1 \right)$ , where  $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{K}}$ , and  $\vec{\mathbf{K}} = \vec{\mathbf{L}} + \vec{\mathbf{T}}$ . The commuting operators are  $\vec{\mathbf{L}}^2$ ,  $\vec{\mathbf{J}}^2$ ,  $\tilde{\mathbf{L}}_3$  and  $\mathbf{J}_3$ , such that the eigenvalues of  $\hat{\mathcal{M}}_1^{vec}$  and their degeneracies read  $\omega^2 = 2\{l(l+1) + j(j+1) - 1\}$  and  $d = \nu(2l+1)(2j+1)$ , respectively. Here  $\nu$  is the degeneracy factor associated with the several possibilities to obtain a given value of  $j$  for a given value of  $l$ .

To find  $\nu$ , let us fix a value  $l \geq 2$ .  $\vec{\mathbf{K}}$  is the sum of the two unit angular momenta, such that  $K = 0, 1, 2$ . For  $K = 2$  the possible values of  $j$  are  $l-2, l-1, \dots, l+2$ , for  $K = 1$  one obtains  $j = l-1, l, l+1$  and for  $K = 0$  one has  $j = l$ . The values  $j = l \pm 2$  therefore appear only when  $K = 2$ , the values  $j = l \pm 1$  are possible when  $K = 1$  or  $K = 2$ , and there are three different ways to get  $j = l$ , that is, when  $K = 0, 1, 2$ . Thus, if we write  $j = l + \sigma$ , where  $\sigma = 0, \pm 1, \pm 2$ , the degeneracies  $\nu$  are as follows:  $\nu = 3$  for  $\sigma = 0$ ,  $\nu = 2$  for  $\sigma = \pm 1$  and  $\nu = 1$  for  $\sigma = \pm 2$ . As a result, the eigenvalues of  $\hat{\mathcal{M}}_1^{vec}$  and their degeneracies can be represented as

$$\omega^2 = (2l + \sigma + 1)^2 + \sigma^2 - 3, \quad d = (3 - |\sigma|)(2l + 1)(2l + 2\sigma + 1), \quad l \geq 2, \quad \sigma = 0, \pm 1, \pm 2. \quad (5.15)$$

For  $l < 2$ , the factor  $\nu$  changes. For instance for  $l = 0$  one has  $j = K$ , such that  $\nu = 1$  for  $\sigma = 2, 1, 0$  and  $\nu = 0$  for  $\sigma = -1, -2$ . Similarly for  $l = 1/2$  one obtains  $\nu = \{1, 2, 2, 0, 0\}$  for  $\sigma = \{2, 1, 0, -1, -2\}$ , respectively. For  $l = 1$  one has  $\nu = \{1, 2, 3, 1, 0\}$  and for  $l = 3/2$  the result is  $\nu = \{1, 2, 3, 2, 0\}$ .

Introducing the new quantum number,  $n = 2l + \sigma + 1$ , the whole spectrum can be given in the following compact form:

$$\hat{\mathcal{M}}_1^{vec} : \quad \omega^2 = n^2 + \sigma^2 - 3, \quad d = \nu(n^2 - \sigma^2), \quad \sigma = 0, \pm 1, \pm 2; \quad (5.16)$$

where  $\nu = 3 - |\sigma|$  for  $n \geq 3$ , and  $\nu = n\delta_{0\sigma} + \delta_{n2}\delta_{1|\sigma|}$  for  $n = 1, 2$ . For the other fluctuation operators one similarly obtains

$$\begin{aligned} \hat{\mathcal{M}}_1^{scal} : \quad \omega^2 &= n^2 + \sigma^2 - 3, \quad d = n^2 - \sigma^2, \quad \sigma = 0, \pm 1, \quad n \geq 2; \\ \hat{\mathcal{M}}_0^{vec} : \quad \omega^2 &= n^2 + \sigma^2 - 1, \quad d = 3(n^2 - \sigma^2), \quad \sigma = 0, \pm 1, \quad n \geq 2; \\ \hat{\mathcal{M}}_0^{scal} : \quad \omega^2 &= n^2 - 1, \quad d = 3n^2, \quad n \geq 1. \end{aligned} \quad (5.17)$$

All these eigenvalues are positive for  $n \geq 2$ . When  $n = 1$ , the operator  $\hat{\mathcal{M}}_1^{vec}$  has one negative mode with eigenvalue  $\omega_-^2 = -2$ . It is worth noting that the sphaleron does not possess any zero modes. This can be understood as follows: the sphaleron solution completely shares the symmetries of the 3-space – both are  $SO(4)$  symmetric. The sphaleron is therefore invariant under spatial translations and rotations. In this case, all zero modes must be of pure gauge origin, but the gauge is completely fixed.

For  $n = 1$ , the vacuum operator  $\hat{\mathcal{M}}_0^{scal}$  has three zero modes. As is seen from (5.14), their eigenfunctions are just constants discussed in the previous section:  $\xi_p^\mu = \delta_p^\mu \tau^p / 2$ . Since the spectra of  $\hat{\mathcal{M}}^{scal}$  and  $\hat{\mathcal{M}}^{FP}$  coincide (but these operators act in the different spaces),  $\hat{\mathcal{M}}_0^{FP}$  also has three constant zero modes,  $\alpha_p = \tau^p / 2$ , which have been discussed above.

The next step is to pick up the physical modes. The partition functions  $Z_j$  specified by (4.16) and (4.23) involve the ratios of the determinants, such that some of the eigenvalues cancel. Notice that (5.6) implies that  $Det'(\hat{M}) = Det'(\hat{M}^{scal})Det'(\hat{M}^{vec})$ , whereas  $Det'(\hat{M}^{scal}) = Det'(\hat{M}^{FP})$ . The partition functions therefore are [25]

$$Z_1 = \exp(-S_E[A^{(sp)}]) \frac{\mathbf{a}}{2i\sqrt{|\omega_-|}} \sqrt{\frac{Det'(\hat{M}_1^{scal}/\mathbf{a}^2)}{Det'(\hat{M}_1^{vec}/\mathbf{a}^2)}}, \quad Z_0 = \frac{4\pi}{\mathbf{a}^3\beta^3} \sqrt{\frac{Det'(\hat{M}_0^{scal}/\mathbf{a}^2)}{Det'(\hat{M}_0^{vec}/\mathbf{a}^2)}}. \quad (5.18)$$

Thus the physical oscillator modes constitute the part of the spectrum of  $\hat{M}^{vec}$  that remains after the subtraction of all eigenvalues of  $\hat{M}^{scal}$ . Since all eigenvalues of  $\hat{M}^{scal}$  are contained in the spectrum of  $\hat{M}^{vec}$  (besides the zero modes of  $\hat{M}_0^{vec}$ ), such a subtraction results in the change of the degeneracy factors of the eigenmodes of  $\hat{M}^{vec}$ . Using (5.16), (5.17), one obtains the eigenvalues and the degeneracies,  $\{\omega^2, d\}$ , of the physical oscillators:

$$\hat{\mathcal{M}}_1^{vec}/\hat{\mathcal{M}}_1^{scal} : \quad \{-2, 1\}, \quad \{1, 4\}, \quad \{n^2 + \sigma^2 - 3, 2(n^2 - \sigma^2)\}, \quad n \geq 3, \quad \sigma = 0, 1, 2; \quad (5.19)$$

also

$$\hat{\mathcal{M}}_0^{vec}/\hat{\mathcal{M}}_0^{scal} : \quad \{0, -3\}, \quad \{n^2, 6(n^2 - 1)\}, \quad n \geq 2. \quad (5.20)$$

Notice that  $\sigma$  in (5.19) assumes three values which correspond to the three “colours” of the “gluon”, and the coefficient 2 in the degeneracy factor refers to its two polarizations, whereas in (5.20) all  $(2 \text{ spin}) \times (3 \text{ isospin})$  gluon degrees of freedom are absorbed by the coefficient 6 in the degeneracy factor. Taking into account the temporal part of the fluctuation operators,  $-\frac{\partial^2}{\partial \tau^2}$ , whose eigenvalues are the Matsubara frequencies,  $4\pi^2 l^2 / \beta^2$ ,

we obtain the following contribution of each physical oscillator,  $\{\omega^2, d\}$ , into the partition function:

$$\prod_{l=-\infty}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} + \omega^2 \right) \right\}^{-d/2}. \quad (5.21)$$

For  $\omega^2 < 0$  or  $\omega^2 = 0$  the term with  $l = 0$  in this product should be omitted, since the corresponding negative or zero modes have already been taken into account. Notice that the negative degeneracy for the zero mode in (5.20) arises simply because this mode is not in the spectrum of  $\hat{\mathcal{M}}_0^{vec}$  and only in that of  $\hat{\mathcal{M}}_0^{scal}$ .

We are now in a position to give the formal closed expressions for the partition functions. For the fluctuations around the sphaleron we obtain

$$\begin{aligned} \frac{\exp(-S_E[A^{(sp)}])}{Z_1} &= i2 \frac{\sqrt{2}}{\mathbf{a}} \prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} - 2 \right) \right\} \times \prod_{l=-\infty}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} + 1 \right) \right\}^2 \times \\ &\times \prod_{l=-\infty}^{\infty} \prod_{\sigma=0,1,2} \prod_{n=3}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} + n^2 + \sigma^2 - 3 \right) \right\}^{n^2 - \sigma^2}, \end{aligned} \quad (5.22)$$

and for the vacuum

$$\frac{1}{Z_0} = \frac{\mathbf{a}^3 \beta^3}{4\pi} \prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} \right) \right\}^{-3} \times \prod_{l=-\infty}^{\infty} \prod_{n=2}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} + n^2 \right) \right\}^{3(n^2-1)}. \quad (5.23)$$

## 6 Evaluation of determinants and the transition rate

We use the zeta function techniques (see [30]-[37], and references therein) to regularize and evaluate the infinite products entering Eqs.(5.22), (5.23). The key steps of our analysis are presented in this section, whereas a large number of the technical details are given in the Appendix.

The basic zeta function relation reads

$$\prod_n \left( \frac{\lambda_n}{\mu} \right)^{d_n} = \exp\{-\zeta'(0) - \ln \mu \zeta(0)\}, \quad (6.1)$$

where

$$\zeta(s) = \sum_n \frac{d_n}{(\lambda_n)^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_n d_n \exp(-t\lambda_n) dt, \quad (6.2)$$

which in fact should be regarded as the definition of the product.

We start by applying this to the zero mode contribution in (5.23):

$$\prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} \right) \right\}^{-3} = \prod_{l=1}^{\infty} \left( \frac{2\pi l}{\mathbf{a}\beta} \right)^{-6}. \quad (6.3)$$

The scale factor  $\mu$  is given by  $\mu = \mathbf{a}\beta/2\pi$ , and the zeta function is

$$\zeta_0(s) = -6 \sum_{l=1}^{\infty} \frac{1}{l^s} = -6\zeta_R(s), \quad \Rightarrow \quad \zeta_0(0) = 3, \quad \zeta'_0(0) = 3 \ln 2\pi, \quad (6.4)$$

where  $\zeta_R(s)$  is the Riemann zeta function. This gives

$$\prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} \right) \right\}^{-3} = \frac{1}{\mathbf{a}^3 \beta^3}, \quad (6.5)$$

such that the overall contribution of the zero modes together with their Matsubara excitations into (5.23) is

$$\frac{\mathbf{a}^3 \beta^3}{4\pi} \prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} \right) \right\}^{-3} = \frac{1}{4\pi}. \quad (6.6)$$

Next, consider the product in (5.22) due to the negative mode. The corresponding zeta function is

$$\zeta_{-}(s) = \sum_{l=1}^{\infty} \left( l^2 - \frac{\beta^2}{2\pi^2} \right)^{-s}, \Rightarrow \zeta_{-}(0) = -\frac{1}{2}, \quad \zeta'_{-}(0) = -\ln \left( \frac{2\sqrt{2}\pi}{\beta} \sin\left(\frac{\beta}{\sqrt{2}}\right) \right) \quad (6.7)$$

(see Appendix, Eq.(A.18)). The normalization factor is  $\mu = (\beta\mathbf{a}/2\pi)^2$ , which yields

$$2 \frac{\sqrt{2}}{\mathbf{a}} \prod_{l=1}^{\infty} \left\{ \frac{1}{\mathbf{a}^2} \left( \frac{4\pi^2 l^2}{\beta^2} - 2 \right) \right\} = 4 \sin \frac{\beta}{\sqrt{2}}. \quad (6.8)$$

Now we want to take into account the contribution of the positive field modes. We introduce the spatial zeta function associated with the positive physical oscillators (5.19), (5.20)

$$\zeta_{spat}(s) = 2 + \sum_{\sigma=0,1,2} \sum_{n=3}^{\infty} \frac{n^2 - \sigma^2}{(n^2 + \sigma^2 - 3)^s} - 3 \sum_{n=2}^{\infty} \frac{n^2 - 1}{(n^2)^s}, \quad (6.9)$$

such that

$$\sqrt{\frac{Det' \hat{\mathcal{M}}_1^{scal}}{Det' \hat{\mathcal{M}}_1^{vec}} \frac{Det' \hat{\mathcal{M}}_0^{vec}}{Det' \hat{\mathcal{M}}_0^{scal}}} = \exp\{\zeta'_{spat}(0)\}. \quad (6.10)$$

One has

$$\zeta_{spat}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \Theta_{spat}(t), \quad (6.11)$$

where the heat kernel is

$$\Theta_{spat}(t) = 2 \exp\{-t\} + \sum_{\sigma=0,1,2} \sum_{n=3}^{\infty} (n^2 - \sigma^2) \exp\{-t(n^2 + \sigma^2 - 3)\} - 3 \sum_{n=2}^{\infty} (n^2 - 1) \exp\{-tn^2\}. \quad (6.12)$$

We shall need the asymptotic expansion of this function for small  $t$

$$\Theta_{spat}(t) \sim \frac{1}{(4\pi t)^{3/2}} \sum_{r=0,1/2,1,\dots} C_r t^r. \quad (6.13)$$

The computation of the coefficients  $C_r$  for  $r \leq 2$  is performed in the Appendix:

$$C_0 = C_{1/2} = C_1 = 0, \quad C_{3/2} = -2(4\pi)^{3/2}, \quad C_2 = 22\pi^2. \quad (6.14)$$

Next, we introduce the thermal zeta function related to the spatial zeta function

$$\zeta_\beta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{l=-\infty}^\infty \exp \left\{ - \left( \frac{2\pi l}{\beta} \right)^2 t \right\} \Theta_{spat}(t) dt. \quad (6.15)$$

As a result, we can collect all parts together and represent the expression for the sphaleron transition rate in the following form:

$$\Gamma = -\frac{\omega_-}{\pi} \frac{\text{Im} Z_1}{Z_0} = \frac{1}{8\sqrt{2}\pi^2 \sin(\beta/\sqrt{2})} \exp \left\{ -\frac{3\pi^2}{g^2} \beta + \zeta'_\beta(0) + \ln \mathbf{a}^2 \zeta_\beta(0) \right\}. \quad (6.16)$$

Here the prefactor on the right hand side includes the contribution of the zero modes and the negative mode. The exponent contains the one-loop contribution of the positive oscillator modes, and the classical sphaleron action

$$S_E[A^{(sp)}] = \frac{3\pi^2}{g^2} \beta. \quad (6.17)$$

To compute the quantities  $\zeta'_\beta(0)$  and  $\zeta_\beta(0)$  entering Eq.(6.16), we represent the heat kernel in (6.12) symbolically as

$$\Theta_{spat}(t) = \sum_\omega \exp\{-\omega^2 t\}, \quad \omega^2 > 0. \quad (6.18)$$

Then the values of  $\zeta_\beta(0)$  and  $\zeta'_\beta(0)$  are given by (see Appendix)

$$\zeta_\beta(0) = \frac{C_2 \beta}{16\pi^2}, \quad \zeta'_\beta(0) = \left\{ \frac{(2 - 2 \ln 2) C_2}{16\pi^2} - \text{PP} \zeta_{spat}\left(-\frac{1}{2}\right) \right\} \beta - 2 \sum_\omega \ln(1 - e^{-\beta \omega}), \quad (6.19)$$

where the coefficient  $C_2$  is specified by (6.14).

All this allows us to represent the transition rate (6.16) as

$$\Gamma = \frac{1}{8\sqrt{2}\pi^2 \sin(\beta/\sqrt{2})} \exp \left\{ -\frac{3\pi^2}{g^2(\mathbf{a})} \beta - \mathcal{E}_0 \beta - \beta(F_1 - F_0) \right\}. \quad (6.20)$$

In this expression, the renormalized gauge coupling constant is

$$\frac{1}{g^2(\mathbf{a})} = \frac{1}{g^2(\mathbf{a}_0)} - \frac{11}{12\pi^2} \ln \left( \frac{\mathbf{a}}{\mathbf{a}_0} \right). \quad (6.21)$$

Here we have returned to the dimensionful  $\mathbf{a}$  and replaced  $g$  by  $g(\mathbf{a}_0)$ , where  $\mathbf{a}_0 = 1/\mu_0$ . This expression agrees with the renormalization group flow, such that it does not depend on the scale  $\mathbf{a}_0$  if  $g(\mathbf{a}_0)$  is chosen to obey the Gell-Mann-Low equation. To fix the scale, we assume that the value of  $g(\mathbf{a}_0)$  is determined by the typical energy of the physical processes in the universe, that is by the physical temperature  $T(\mathbf{a}_0) = 1/\beta \mathbf{a}_0$ . Then we use the QCD data (see for example [29])

$$T(\mathbf{a}_0) = 100 \text{ GeV}, \quad \frac{g^2(\mathbf{a}_0)}{4\pi} = 0.12, \quad (6.22)$$

and assume that the weak coupling region extends up to some  $\mathbf{a}_{max}$ . One can choose  $\mathbf{a}_{max} \sim 10 \div 100 \mathbf{a}_0$ .



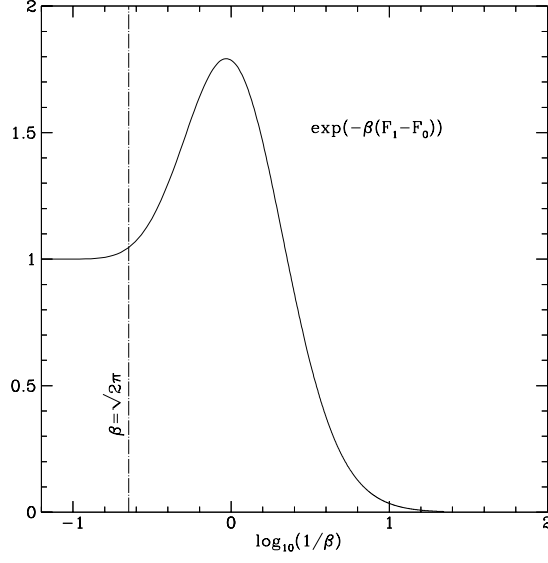


Figure 1: The thermal function  $\exp(-\beta(F_1 - F_0))$ .

$\mathcal{E}_0$  is the contribution of the zero field oscillations, that is, the Casimir energy,

$$\mathcal{E}_0 = \text{PP}\zeta_{\text{spat}}(-\frac{1}{2}) + \frac{11}{4}(\ln 2 - 1). \quad (6.23)$$

It is worth noting that this quantity can be computed exactly in this case. The corresponding computation itself presents some methodological interest and is given in the Appendix. The result is

$$\begin{aligned} \mathcal{E}_0 = & \frac{5}{6} + \frac{11}{4}(\ln 2 - 1 - \gamma) + \int_0^1 dz \sqrt{1 - z^2} \times \\ & \times \int_0^1 dt \tan\left(\frac{\pi t}{2}\right) \left\{ (z^2 + 4)F(z, t) + (4z^2 - 2)G(z, t, \sqrt{2}) + 9z^2 G(z, t, \sqrt{3}) \right\}, \end{aligned} \quad (6.24)$$

where

$$F(z, t) = t - \frac{\sinh(\pi z t)}{\sinh(\pi z)}, \quad G(z, t, q) = t - \frac{\sin(\pi q z t)}{\sin(\pi q z)} + \frac{2}{\pi} \frac{\sin(\pi t)}{(1 - q^2 z^2)}. \quad (6.25)$$

The numerical value is

$$\mathcal{E}_0 = -1.084 \dots \quad (6.26)$$

The contribution of the thermal degrees of freedom in (6.20) is

$$\begin{aligned} \beta(F_1 - F_0) = & 4 \ln(1 - e^{-\beta}) + 2 \sum_{\sigma=0,1,2} \sum_{n=3}^{\infty} (n^2 - \sigma^2) \ln(1 - e^{-\beta \sqrt{n^2 + \sigma^2 - 3}}) - \\ & - 6 \sum_{n=2}^{\infty} (n^2 - 1) \ln(1 - e^{-\beta n}), \end{aligned} \quad (6.27)$$

and the remaining sums in this expression can be evaluated numerically (see Fig.1).

One can see that  $F_1$  and  $F_0$  are precisely the free energies of the physical oscillators (5.19), (5.20). Altogether Eqs.(6.20)-(6.27) provide the desired solution of the one-loop

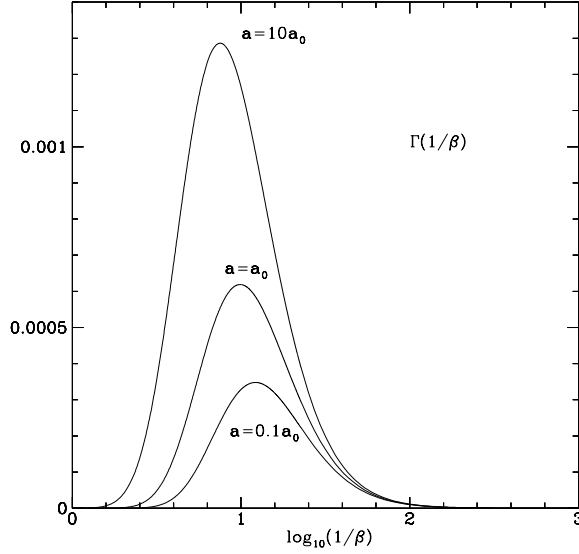


Figure 2: The sphaleron transition rate  $\Gamma(1/\beta)$ .

sphaleron transition problem. The numerical curves of  $\Gamma(1/\beta)$  evaluated according to these formulas for several values of  $\mathbf{a}$  are presented in Fig.2.

This solution makes sense under the following assumptions:

$$\mathbf{a} \leq \mathbf{a}_{max}, \quad \frac{1}{\sqrt{2\pi}} < \frac{1}{\beta} \ll \frac{3\pi^2}{g^2(\mathbf{a})}. \quad (6.28)$$

The first condition is the weak coupling requirement. When the scale factor  $\mathbf{a}$  is too large, the running coupling constant (6.21) becomes big (confinement phase), and the effects of the strong coupling can completely change the semiclassical picture. That is why our solution can be trusted only for small values of the size of the universe. The other condition in (6.28) requires that the thermal fluctuations are small compared to the classical sphaleron energy, such that the perturbation theory is valid. Notice that each curve in Fig.2 develops a maximum at some temperature  $1/\beta_{max}(\mathbf{a})$ ; however, this value seems to be already beyond the scope of the approximation:  $3\pi^2/g^2(\mathbf{a}) \sim 2/\beta_{max}(\mathbf{a})$ . The subsequent decrease of  $\Gamma$  is presumably fictitious. Indeed, it is natural to expect that the transition rate is increasing with growing temperature [12]. Thus, our results can be trusted at best only for  $1/\beta < 1/\beta_{max}(\mathbf{a})$ .

One can also find the high temperature limit for the solution (but the upper bound in (6.28) is to be assumed). To determine the asymptotic behavior of the free energy, the procedure is the following [31]. First, one returns to the zeta function  $\zeta_\beta(s)$  and replaces in (6.15) the heat kernel  $\Theta_{spat}(t)$  by its asymptotic expansion (6.13). Then one takes the integral over  $t$ , and the sum over  $l$  reduces to the Riemann zeta function. As a result, one arrives at the following asymptotic expansion for small  $\beta$  [31]:

$$\begin{aligned} \zeta'_\beta(0) = & \frac{\pi^2}{45} \frac{C_0}{\beta^3} + \frac{\zeta(3)}{2\pi\sqrt{\pi}} \frac{C_{1/2}}{\beta^2} + \frac{C_1}{12\beta} - \frac{C_{3/2}}{4\pi\sqrt{\pi}} \ln \beta + \frac{C_2}{8\pi^2} \left( \gamma + \ln \frac{\beta}{4\pi} \right) \beta + \\ & + \frac{1}{4\pi\sqrt{\pi}} \sum_{r \geq 1} C_{r+3/2} \left( \frac{\beta}{2\pi} \right)^{2r} \zeta_R(2r) \Gamma(r) + \zeta'_{spat}(0). \end{aligned} \quad (6.29)$$

This gives the free energy (see Eq.(6.19))

$$F = -\frac{1}{\beta}\zeta'_\beta(0) + \frac{(1 - \ln 2)C_2}{8\pi^2} - \text{PP}\zeta_{\text{spat}}(-\frac{1}{2}). \quad (6.30)$$

As an illustration we apply this to the vacuum fluctuations alone. The corresponding spatial zeta function is given by the last piece in Eq.(6.9):  $\zeta_{\text{spat}}^0(s) = 3\zeta_R(2s-2) - 3\zeta_R(2s)$ , its coefficients  $C_r$  are computed in the Appendix (one has  $C_2 = 0$ ), and the result is

$$F_0 = -2\pi^2 \left( \frac{\pi^2}{15\beta^4} - \frac{1}{2\beta^2} - \frac{3}{2\pi^2} \frac{\ln \beta}{\beta} + \frac{3}{4\pi^2\beta} + \frac{11}{80\pi^2} + O(\beta) \right). \quad (6.31)$$

Here  $2\pi^2$  is the volume of  $S^3$  (remember that  $\beta = 1/\mathbf{a}T$ ), and the leading term is just the free energy of a gas of noninteracting, massless particles with  $2 \times 3$  polarization states.

Now, let us return to the full expression (6.9) for  $\zeta_{\text{spat}}(s)$ . The values of  $C_r$  in this case are given by (6.14). Using (6.23), (6.30) one obtains

$$-\beta(F_1 - F_0) = 4\ln \beta + \zeta'_{\text{spat}}(0) + \left( \frac{11}{4}(\gamma + \ln \frac{\beta}{4\pi}) + \mathcal{E}_0 \right) \beta + O(\beta^2). \quad (6.32)$$

The quantity  $\zeta'_{\text{spat}}(0)$  is computed in the Appendix:

$$\kappa \equiv \exp\{\zeta'_{\text{spat}}(0)\} = \frac{2\sqrt{2}}{\pi^3} \sinh^4(\pi) |\sin(\sqrt{2}\pi)| \exp\{\mathcal{J}(1) - \mathcal{I}(\sqrt{2}) - \mathcal{I}(\sqrt{3})\}, \quad (6.33)$$

where

$$\mathcal{J}(x) = \pi \int_0^x t^2 \coth(\pi t) dt, \quad \mathcal{I}(x) = \int_0^x \left( \frac{2t}{1-t^2} + \pi t^2 \cot(\pi t) \right) dt, \quad (6.34)$$

with the numerical value

$$\kappa = 1250.21 \dots \quad (6.35)$$

(It is interesting to observe the large value of  $\kappa$ . The corresponding quantity for the electroweak sphaleron is suppressed by several orders of magnitude [7], [8].)

Finally, one obtains (see Fig.3)

$$\exp\{-\beta(F_1 - F_0)\} = \kappa \beta^4 \left( 1 + \frac{11}{4} \beta \ln \beta \right) + O(\beta^5) \quad \text{as } \beta \rightarrow 0. \quad (6.36)$$

As a result, we arrive at the following expression for the sphaleron transition rate in the high temperature limit:

$$\Gamma(\beta) = \frac{\kappa \beta^3}{8\pi^2} \exp \left\{ -\frac{3\pi^2}{g^2(\mathbf{a})} \beta \right\}, \quad (6.37)$$

where we have neglected also the Casimir term. As one can see from Fig.3, the high temperature approximation can be reasonable for  $1/\beta \geq 10^3$ . On the other hand, the temperature should be less than the sphaleron energy  $3\pi^2/g^2(\mathbf{a})$ . These two conditions imply that (6.37) makes sense only for small  $g(\mathbf{a})$ :  $10^3 \leq 3\pi^2/g^2(\mathbf{a})$ , that is, for small  $\mathbf{a}$ .

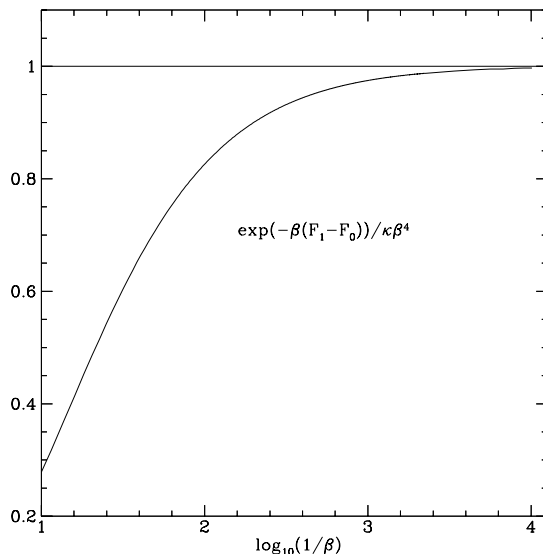


Figure 3: The ratio between the thermal function  $\exp(-\beta(F_1 - F_0))$  and its high temperature asymptote  $1250.21\beta^4$ .

## 7 Concluding remarks

In this paper we have obtained the exact solution of the sphaleron transition problem for a pure non-Abelian gauge field in a static Einstein universe. This has been achieved by the straightforward diagonalization of the one-loop fluctuation operators with the subsequent computation of the functional determinants in the zeta function regularization scheme. To carry out this program, the following points have been crucial: the high symmetry of the sphaleron solution and the fact that the sphaleron configuration consists of the gauge field alone. Actually, these properties of the model under consideration make it somewhat similar to the instanton theory [2], [25], [26]. It is worth noting that the solution obtained in this paper is unique in the sense that no other exact solutions of the sphaleron models in  $3 + 1$  spacetime dimensions are known.

Eqs.(6.20)-(6.27) and (6.37) are our principal results. They specify the number of transitions between the neighboring topological sectors per unit conformal time  $\eta$ . It should be stressed that such transitions do not lead to any violation of chiral fermion number unless the thermal equilibrium between the different topological sectors is broken. This can arise, for instance, when a fermion asymmetry is present. Then one has to introduce a small chemical potential  $\mu$  for the fermions [6], [5]. This favors those transitions which erase the asymmetry. Specifically, let  $\Delta N = N_F - \bar{N}_F$  be the fermion number of the universe, then

$$\frac{d}{d\eta}\Delta N \approx -\frac{\mu}{T}\Gamma. \quad (7.1)$$

Notice that – since  $\Gamma$  defines the number of transitions in the whole universe –  $\Delta N$  refers to the whole universe as well, and not to the unit volume. For one doublet of chiral fermions, standard thermodynamics gives

$$N_F = \frac{V}{\pi^2} \int_0^\infty \frac{p^2 dp}{\exp\{\frac{p-\mu}{T}\} + 1} = \frac{3VT^3}{2\pi^2} \zeta_R(3) + \frac{\mu}{6} VT^2 + O(\mu^2), \quad (7.2)$$

where  $V = 2\pi^2 \mathbf{a}^3$  is the volume of 3-space, such that  $\Delta N = 2\pi^2 \mathbf{a} \mu / 3\beta^2$ . Finally, passing to the physical time  $t = \mathbf{a} \eta$ , one obtains the fermion number diffusion rate

$$\frac{1}{\Delta N} \frac{d}{dt} \Delta N = -\frac{3\beta^3}{2\pi^2 \mathbf{a}} \Gamma\left(\frac{1}{\beta}\right). \quad (7.3)$$

In fact, we do not specify the nature of the fields under consideration. The discussion of the possible applications of the results obtained in this paper will be given separately. At present we just mention where our results can be used. Let us recall the typical values of the parameters  $\mathbf{a}$ ,  $\beta$  and  $T = 1/\mathbf{a}\beta$ . The range of  $T$  is restricted by the condition of the validity of the semiclassical picture:  $T \geq 1 \div 10$  GeV (“deconfinement phase”), whereas the metastability condition requires that the conformal temperature,  $1/\beta$ , is not too high (see Eq.(6.28)). The size of the universe,  $\mathbf{a} = 1/T\beta$ , therefore should not be too large compared to  $1/T$ . Such conditions can be met in the context of the finite volume QCD (the typical volume in that case is  $\mathbf{a}^3 \sim 1 \text{ fm}^3$ ; see [19] and references therein). Another natural possibility relates to the pre-inflation cosmology. In this case, the gravitating Yang-Mills field can arise in the context of a superstring theory [38]. In fact, the semiclassical sphaleron transition picture applies also after inflation. However, the conformal temperature is enormously large then,  $1/\beta = \mathbf{a}T \sim s^{1/3} \sim 10^{28}$ , where  $s$  is the total entropy of the universe. Equivalently, one can say that the temperature  $T$  is huge in comparison with the sphaleron barrier  $3\pi^2/g^2(\mathbf{a})\mathbf{a}$ , such that there is no suppression for transitions between the different topological sectors at all. Unfortunately, there are no reliable methods for computing the transition rate  $\Gamma$  in this limit. One can use a dimensional argument to estimate that the rate related to the unit physical 4-volume,  $\Gamma/\mathbf{a}^4$ , should be proportional to  $T^4$ . Then  $\Gamma(1/\beta) \sim 1/\beta^4$ , and (7.3) yields

$$\frac{1}{\Delta N} \frac{d}{dt} \Delta N \sim T. \quad (7.4)$$

This agrees with the usual estimate for the fermion number dissipation rate at very high temperatures [4].

## Acknowledgments

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## Appendix. Zeta function techniques

In Sections **A–D** below the detailed computation of  $\text{PP}\zeta_{\text{spat}}(-1/2)$  and  $\zeta'_{\text{spat}}(0)$  is presented. The basic idea is to expand these quantities into certain series of the Riemann zeta functions and then to use the appropriate summation formulae. The asymptotic expansion for the heat kernel  $\Theta_{\text{spat}}(t)$  is derived in Sec.**E**. Section **F** contains the computation of  $\zeta_\beta(0)$  and  $\zeta'_\beta(0)$  for a thermal system.

### A. Summation formulae

Consider the generating function for the Bernoulli polynomials (see [37], p.804)

$$\frac{xe^{xt}}{e^x - 1} = \sum_{k=0}^{\infty} B_k(t) \frac{x^k}{k!}, \quad |x| < 2\pi. \quad (\text{A.1})$$

Putting here  $t = 0$  one obtains

$$\coth(x) = \frac{1}{x} \sum_{k=0}^{\infty} B_{2k} \frac{(2x)^{2k}}{(2k)!}, \quad |x| < \pi, \quad (\text{A.2})$$

where  $B_k(0) = B_k$  are the Bernoulli numbers. Using the relation between the Bernoulli numbers and the Riemann zeta function (see [37], p.807),

$$B_{2k} = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta_R(2k), \quad (\text{A.3})$$

and considering the replacement  $x \rightarrow ix$ , one finds

$$\coth(x) = -\frac{2}{x} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k); \quad \cot(x) = -\frac{2}{x} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k). \quad (\text{A.4})$$

The sums on the right hand sides converge only for  $|x| < \pi$ , whereas the left hand sides are meromorphic functions on the whole complex plane. One can therefore consider these relations for any  $x$  as a result of analytic continuation.

Let us now restrict ourselves to the real values of  $x$ . Integrating both sides of (A.4) one obtains

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k) = \ln \frac{x}{\sinh(x)}; \quad \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k) = \ln \frac{x}{\sin(x)}. \quad (\text{A.5})$$

In the second of these formulae one should assume that  $x < \pi$ , unless the integration rule for the poles of  $\cot(x)$  is specified (see below).

Multiplying (A.4) by  $x^2$  and integrating from zero to  $x$  one obtains

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k-2) = \mathcal{J}\left(\frac{x}{\pi}\right), \quad \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x}{\pi}\right)^{2k} \zeta_R(2k-2) = \tilde{\mathcal{J}}\left(\frac{x}{\pi}\right), \quad (\text{A.6})$$

where

$$\mathcal{J}(x) = \pi \int_0^x t^2 \coth(\pi t) dt, \quad (\text{any } x); \quad \tilde{\mathcal{J}}(x) = -\pi \int_0^x t^2 \cot(\pi t) dt \quad (x < \pi). \quad (\text{A.7})$$

Eqs.(A.5)-(A.7) will be used below. To proceed further, we return for a moment to Eq.(A.1). Consider the equation which is obtained from (A.1) under the replacement  $x \rightarrow -x$ . Taking the difference of the two equations one obtains

$$\frac{e^{xt}}{e^x - 1} + \frac{e^{-xt}}{e^{-x} - 1} = 2 \sum_{k=0}^{\infty} B_{2k+1}(t) \frac{x^{2k}}{(2k+1)!}. \quad (\text{A.8})$$

Using the explicit form of the  $k = 0$  term on the right hand side,  $2B_1(t) = 2t - 1$ , and utilizing also the following relation (see [37], p.807):

$$\zeta_R(2k+1) = (-1)^{k+1} \frac{(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt, \quad (\text{A.9})$$

one finds

$$\sum_{k=1}^{\infty} (-1)^k \left( \frac{x}{2\pi} \right)^{2k} \zeta_R(2k+1) = -\frac{\pi}{2} \int_0^1 \left( 1 - 2t + \frac{e^{xt}}{e^x - 1} + \frac{e^{-xt}}{e^{-x} - 1} \right) \cot(\pi t) dt. \quad (\text{A.10})$$

Replacing here  $x \rightarrow 2x$  and  $t \rightarrow 2t - 1$ , and considering also  $x \rightarrow ix$  one arrives at

$$\sum_{k=1}^{\infty} (-1)^k \left( \frac{x}{\pi} \right)^{2k} \zeta_R(2k+1) = -\frac{\pi}{2} I(x), \quad \sum_{k=1}^{\infty} \left( \frac{x}{\pi} \right)^{2k} \zeta_R(2k+1) = -\frac{\pi}{2} \tilde{I}(x), \quad (\text{A.11})$$

where

$$I(x) = \int_0^1 \left( t - \frac{\sinh(xt)}{\sinh(x)} \right) \tan\left(\frac{\pi t}{2}\right) dt, \quad \tilde{I}(x) = \int_0^1 \left( t - \frac{\sin(xt)}{\sin(x)} \right) \tan\left(\frac{\pi t}{2}\right) dt. \quad (\text{A.12})$$

Next, one deduces from (A.11) that

$$\sum_{k=1}^{\infty} (-1)^k \nu^{2k} z^{2k} \zeta_R(2k+1) = -\frac{\pi}{2} I(\pi \nu z). \quad (\text{A.13})$$

Using

$$\frac{\Gamma(k+1/2)}{\Gamma(k+2)} = \frac{4}{\sqrt{\pi}} \int_0^1 dz z^{2k} \sqrt{1-z^2}, \quad \frac{\Gamma(k+1/2)}{\Gamma(k+3)} = \frac{8}{3\sqrt{\pi}} \int_0^1 dz z^{2k} (1-z^2)^{3/2}, \quad (\text{A.14})$$

we finally obtain the following formulae:

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^k \nu^{2k} \frac{\Gamma(k+1/2)}{\Gamma(k+2)} \zeta_R(2k+1) &= -2\sqrt{\pi} \int_0^1 dz \sqrt{1-z^2} I(\pi \nu z), \\ \sum_{k=1}^{\infty} (-1)^k \nu^{2k} \frac{\Gamma(k+1/2)}{\Gamma(k+3)} \zeta_R(2k+1) &= -\frac{4}{3}\sqrt{\pi} \int_0^1 dz (1-z^2)^{3/2} I(\pi \nu z), \end{aligned} \quad (\text{A.15})$$

together with the corresponding two relations obtained by  $\nu \rightarrow i\nu$ ,  $I(\pi \nu z) \rightarrow \tilde{I}(\pi \nu z)$ . Here the integrals can be considered for arbitrary values of  $\nu$ , which defines the analytic extension of the series.

## B. The basic zeta function relation

Consider the following zeta function

$$\zeta_{\nu}^0(s) = \sum_{n=1}^{\infty} \frac{1}{(n^2 + \nu^2)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{n=1}^{\infty} \exp\{-t(n^2 + \nu^2)\} dt. \quad (\text{A.16})$$

In order to express this function in terms of the Riemann zeta function, we expand  $\exp(-t\nu^2) = \sum_k (-1)^k \nu^{2k} t^k / k!$ , and perform the integration over  $t$ . Then the sum over  $n$  gives the Riemann zeta function, such that

$$\zeta_{\nu}^0(s) = \zeta_R(2s) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \nu^{2k} \frac{\Gamma(k+s)}{\Gamma(s)} \zeta_R(2k+2s). \quad (\text{A.17})$$

This relation will turn out to be especially handy. Taking the pole of the gamma function at  $s = 0$  into account, using Eq.(A.5) and utilizing  $\zeta_R(0) = -\frac{1}{2}$ ,  $\zeta'_R(0) = -\frac{1}{2} \ln 2\pi$ , one finds

$$\zeta_{\nu}^0(0) = -\frac{1}{2}, \quad \frac{d}{ds} \zeta_{\nu}^0(0) = -\ln(2\pi) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \nu^{2k} \zeta_R(2k) = \ln \frac{\nu}{2 \sinh(\pi \nu)}. \quad (\text{A.18})$$

For a harmonical oscillator, for instance, one has

$$\zeta(s) = \sum_{l=1}^{\infty} \left\{ \left( \frac{2\pi l}{\beta} \right)^2 + \omega^2 \right\}^{-s} = \left( \frac{\beta}{2\pi} \right)^{2s} \zeta_{\beta\omega/2\pi}^0(s) \Rightarrow \frac{d}{ds} \zeta(0) = \ln \frac{\omega}{2 \sinh(\beta\omega/2)}, \quad (\text{A.19})$$

which gives rise to the formula (6.8) in the main text when  $\omega = i\sqrt{2}$ .

We shall also use another zeta function

$$\zeta_{\nu}^2(s) = \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + \nu^2)^s} = \zeta_R(2s-2) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \nu^{2k} \frac{\Gamma(k+s)}{\Gamma(s)} \zeta_R(2k+2s-2). \quad (\text{A.20})$$

Using Eqs.(A.6) and (A.7) one obtains

$$\zeta_{\nu}^2(0) = 0, \quad \frac{d}{ds} \zeta_{\nu}^2(0) = 2\zeta'_R(-2) + \mathcal{J}(\nu), \quad (\text{A.21})$$

where  $\mathcal{J}(\nu)$  is defined by (A.7).

### C. Evaluation of $\text{PP}\zeta_{\text{spat}}(-\frac{1}{2})$

We consider the spatial zeta function

$$\zeta_{\text{spat}}(s) = 2 + \sum_{\sigma=0,1,2} \sum_{n=3}^{\infty} \frac{n^2 - \sigma^2}{(n^2 + \sigma^2 - 3)^s} - 3 \sum_{n=2}^{\infty} \frac{n^2 - 1}{(n^2)^s}, \quad (\text{A.22})$$

which can be represented in the form

$$\zeta_{\text{spat}}(s) = \mathcal{A}(s) + \mathcal{B}(s), \quad (\text{A.23})$$

where

$$\mathcal{A}(s) = 2 + 3\zeta_R(2s) - 3\zeta_R(2s-2) + \sum_{\nu^2} (\nu^2 - 1) (4 + \nu^2)^{-s}, \quad (\text{A.24})$$

and

$$\mathcal{B}(s) = \sum_{\nu^2} \left\{ \zeta_{\nu}^0(s-1) - (1 + \nu^2)^{1-s} \right\} - \sum_{\nu^2} (3 + 2\nu^2) \left\{ \zeta_{\nu}^0(s) - (1 + \nu^2)^{-s} \right\}. \quad (\text{A.25})$$

Here  $\zeta_{\nu}^0(s)$  is defined by Eq.(A.16), and  $\nu^2 = \sigma^2 - 3 = 1, -2, -3$ . In  $\mathcal{A}(s)$  one can simply put  $s = -1/2$ , which yields  $\text{PP}\mathcal{A}(-1/2) = \mathcal{A}(-1/2) = -91/40 - 3\sqrt{2}$ .

Let us now consider  $\mathcal{B}(s-1/2)$  in the limit  $s \rightarrow 0$ . We first analyze  $\zeta_{\nu}^0(s-1/2)$ . Using Eq.(A.17) we obtain

$$\zeta_{\nu}^0(s-\frac{1}{2}) = \zeta_R(2s-1) - \nu^2 \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s-\frac{1}{2})} \zeta_R(2s+1) + \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \nu^{2k} \frac{\Gamma(k+s-1/2)}{\Gamma(s-\frac{1}{2})} \zeta_R(2k+2s-1). \quad (\text{A.26})$$

Let us now consider the limit where  $s$  tends to zero. Then the second term on the right hand side diverges due to the pole of the Riemann zeta function,

$$\zeta_R(1+s) = \frac{1}{s} + \gamma + O(s^2), \quad (\text{A.27})$$

where  $\gamma$  is the Euler constant. The remaining terms in (A.26) are all finite. The principal part is

$$\text{PP}\zeta_{\nu}^0(-\frac{1}{2}) = \lim_{s \rightarrow 0} \frac{d}{ds} \left( s \zeta_{\nu}^0(s - \frac{1}{2}) \right). \quad (\text{A.28})$$



Using  $\zeta_R(-1) = -1/12$ ,  $\Gamma(-1/2) = -2\sqrt{\pi}$ , and replacing  $k \rightarrow k+1$  in the sum entering Eq.(A.26), one obtains

$$\text{PP}\zeta_\nu^0(-\frac{1}{2}) = -\frac{1}{12} - \frac{1}{2}\nu^2(1-\gamma) + \frac{\nu^2}{2\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^k \nu^{2k} \frac{\Gamma(k+1/2)}{\Gamma(k+2)} \zeta_R(2k+1). \quad (\text{A.29})$$

Finally, taking into account Eqs.(A.15), one arrives at

$$\text{PP}\zeta_\nu^0(-\frac{1}{2}) = -\frac{1}{12} - \frac{1}{2}\nu^2(1-\gamma) - \nu^2 \int_0^1 dz \sqrt{1-z^2} I(\pi\nu z), \quad (\text{A.30})$$

where  $I(x)$  is defined by Eq.(A.12). Similarly, one obtains

$$\text{PP}\zeta_\nu^0(-\frac{3}{2}) = \frac{1}{120} - \frac{1}{8}\nu^2 - \frac{1}{2}\nu^4(1-\frac{3}{4}\gamma) - \nu^4 \int_0^1 dz (1-z^2)^{3/2} I(\pi\nu z). \quad (\text{A.31})$$

The next step is to insert these relations in Eq.(A.25) and to compute the sum over  $\nu^2 = 1, -2, -3$ . The  $\nu^2 = 1$  case presents no problems, whereas the negative values of  $\nu^2$  should be treated with some care. Let us pass in (A.30) to negative  $\nu^2$ ,  $\nu \rightarrow iq$ , where  $q = |q|$ . For  $q < 1$  one obtains

$$\text{PP}\zeta_\nu^0(-\frac{1}{2}) \rightarrow \text{PP}\zeta_{iq}^0(-\frac{1}{2}) = -\frac{1}{12} + \frac{1}{2}q^2(1-\gamma) + q^2 \int_0^1 dz \sqrt{1-z^2} \tilde{I}(\pi q z), \quad (\text{A.32})$$

where  $\tilde{I}(x)$  is defined by Eq.(A.12). Next we consider the following integral representation for  $\sqrt{1-q^2}$  which is valid for  $0 < q < 1$ :

$$\sqrt{1-q^2} = 1 - \frac{2q^2}{\pi} \int_0^1 dz \sqrt{1-z^2} \int_0^1 \frac{\sin(\pi t)}{1-q^2 z^2} \tan\left(\frac{\pi t}{2}\right) dt. \quad (\text{A.33})$$

This implies

$$\begin{aligned} \text{PP}\zeta_{iq}^0(-1/2) - \sqrt{1-q^2} &= -1 - \frac{1}{12} + \frac{1}{2}q^2(1-\gamma) + \\ &+ q^2 \int_0^1 dz \sqrt{1-z^2} \int_0^1 \left( t - \frac{\sin(\pi z q t)}{\sin(\pi z q)} + \frac{2}{\pi} \frac{\sin(\pi t)}{1-q^2 z^2} \right) \tan\left(\frac{\pi t}{2}\right) dt. \end{aligned} \quad (\text{A.34})$$

Here we can safely extend the range of  $q$  from  $0 < q < 1$  to  $0 < q < 2$ , thus taking the values  $q = \sqrt{2}, \sqrt{3}$  into account, that is,  $\nu^2 = -2, -3$ . This allows us to compute the second sum over  $\nu^2$  in the formula (A.25) for  $\text{PP}\mathcal{B}(-1/2)$ . The first sum can be done with a similar rearrangement, using the formula

$$(1-q^2)^{3/2} = 1 - \frac{3}{2}q^2 + \frac{2q^4}{\pi} \int_0^1 dz (1-z^2)^{3/2} \int_0^1 \frac{\sin(\pi t)}{1-q^2 z^2} \tan\left(\frac{\pi t}{2}\right) dt. \quad (\text{A.35})$$

Finally, collecting everything together, we arrive at

$$\text{PP}\zeta_{\text{spat}}(-\frac{1}{2}) = \frac{5}{6} - \frac{11}{4}\gamma + 5\mathcal{R}_1(1) - \mathcal{R}_3(1) + 2\mathcal{P}_1(\sqrt{2}) - 4\mathcal{P}_3(\sqrt{2}) + 9\mathcal{P}_1(\sqrt{3}) - 9\mathcal{P}_3(\sqrt{3}); \quad (\text{A.36})$$

where

$$\begin{aligned} \mathcal{R}_m(\nu) &= \int_0^1 dz (1-z^2)^{m/2} \int_0^1 \left( t - \frac{\sinh(\pi z \nu t)}{\sinh(\pi z \nu)} \right) \tan\left(\frac{\pi t}{2}\right) dt, \\ \mathcal{P}_m(q) &= \int_0^1 dz (1-z^2)^{m/2} \int_0^1 \left( t - \frac{\sin(\pi z q t)}{\sin(\pi z q)} + \frac{2}{\pi} \frac{\sin(\pi t)}{1-q^2 z^2} \right) \tan\left(\frac{\pi t}{2}\right) dt. \end{aligned} \quad (\text{A.37})$$

#### D. Evaluation of $\zeta'_{spat}(0)$

The procedure in this case is similar to that of the previous section. One again starts from Eqs.(A.23)-(A.25), but now, using (A.16) and (A.20), it is convenient to represent the function  $\mathcal{B}(s)$  in the following equivalent form:

$$\mathcal{B}(s) = \sum_{\nu^2} \left\{ \zeta_{\nu}^2(s) - (1 + \nu^2)^{-s} \right\} - \sum_{\nu^2} (3 + \nu^2) \left\{ \zeta_{\nu}^0(s) - (1 + \nu^2)^{-s} \right\}. \quad (\text{A.38})$$

The next steps are straightforward. Using (A.18) and (A.21) one obtains

$$\mathcal{B}'(0) = \sum_{\nu^2} \left\{ 2\zeta'_R(-2) + \mathcal{J}(\nu) + \ln(1 + \nu^2) \right\} - \sum_{\nu^2} (3 + \nu^2) \ln \frac{\nu(1 + \nu^2)}{2 \sinh(\pi\nu)}, \quad (\text{A.39})$$

where  $\mathcal{J}(\nu)$  is defined by (A.7), and  $\nu^2 = 1, -2, -3$ . For negative  $\nu^2 = -q^2 < 0$  one again needs some minor modifications. No changes are needed for the second sum in (A.39):

$$\ln \frac{\nu(1 + \nu^2)}{2 \sinh(\pi\nu)} \rightarrow \ln \frac{q(1 - q^2)}{2 \sin(\pi q)}, \quad (\text{A.40})$$

where we can put  $q = \sqrt{2}, \sqrt{3}$ . In the first sum one has

$$\mathcal{J}(\nu) + \ln(1 + \nu^2) \rightarrow \tilde{\mathcal{J}}(q) + \ln(1 - q^2) = -\pi \int_0^q t^2 \cot(\pi t) dt - \int_0^q \frac{2t dt}{1 - t^2} \equiv -\mathcal{I}(q), \quad (\text{A.41})$$

where

$$\mathcal{I}(q) = \int_0^q \left( \frac{2t}{1 - t^2} + \pi t^2 \cot(\pi t) \right) dt, \quad (\text{A.42})$$

which can also be extended to the values  $q = \sqrt{2}, \sqrt{3}$ . For  $\mathcal{A}'(0)$  one has

$$\mathcal{A}'(0) = -3 \ln(2\pi) - 6\zeta'_R(-2) + 3 \ln 2, \quad (\text{A.43})$$

which finally gives

$$\zeta'_{spat}(0) = \ln \left( \frac{2\sqrt{2}}{\pi^3} \sinh^4(\pi) |\sin(\sqrt{2}\pi)| \right) + \mathcal{J}(1) - \mathcal{I}(\sqrt{2}) - \mathcal{I}(\sqrt{3}), \quad (\text{A.44})$$

with  $\mathcal{J}(x)$  and  $\mathcal{I}(x)$  being defined in Eqs.(A.7) and (A.42), respectively.

#### E. Asymptotic expansion of $\Theta_{spat}(t)$

Let us consider the following function

$$\Theta(i, j, k|t) = \sum_{n=i}^{\infty} (n^2 - j) e^{-(n^2+k)t} = -e^{-kt} \left( \frac{d}{dt} + j \right) \sum_{n=i}^{\infty} e^{-n^2 t}. \quad (\text{A.45})$$

We want to find its asymptotic expansion for small  $t$ ,

$$\Theta(i, j, k|t) \sim \frac{1}{(4\pi t)^{3/2}} \sum_{r=0, 1/2, 1, \dots} C_r t^r. \quad (\text{A.46})$$

Such an expansion can be obtained with the use of the theta function identity [36]

$$\sum_{n=-\infty}^{\infty} \exp(-tn^2) = \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} \exp \left\{ -\frac{\pi^2}{t} n^2 \right\}, \quad (\text{A.47})$$

which implies that

$$\sum_{n=i}^{\infty} \exp(-tn^2) \sim \frac{1}{2} \sqrt{\frac{\pi}{t}} - \frac{1}{2} - \sum_{n=1}^{i-1} \exp(-tn^2), \quad (\text{A.48})$$

since for  $t \rightarrow 0$  all terms with  $\exp\{-\pi^2 n^2/t\}$ ,  $n \neq 0$  vanish faster than any power of  $t$  and can therefore be omitted. Inserting this into (A.45) and expanding the remaining exponents we find

$$\Theta(i, j, k|t) \sim \frac{\sqrt{\pi}}{4t^{3/2}} - \frac{\sqrt{\pi}}{4t^{1/2}}(k+2j) + j/2 + \sum_{n=1}^{i-1} (j-n^2) + \frac{\sqrt{\pi}}{8} k(k+4j)t^{1/2} + O(t). \quad (\text{A.49})$$

The coefficients  $C_r$  for  $r \leq 2$  are therefore

$$C_0 = 2\pi^2, \quad C_{1/2} = 0, \quad C_1 = -2\pi^2(k+2j), \quad C_{3/2} = 8\pi\sqrt{\pi}\left(\frac{j}{2} + \sum_{n=1}^{i-1} (j-n^2)\right), \quad C_2 = \pi^2 k(k+4j). \quad (\text{A.50})$$

The heat kernel (6.12) in the main text can be represented as

$$\Theta_{\text{spat}}(t) = 2e^{-t} + \sum_{\sigma=0,1,2} \Theta(3, \sigma^2, \sigma^2 - 3|t) - 3\Theta(2, 1, 0|t), \quad (\text{A.51})$$

which finally gives rise to Eq.(6.14).

## F. Evaluation of $\zeta_\beta(0)$ and $\zeta'_\beta(0)$ [35]

Consider the thermal zeta function for an arbitrary system of harmonic oscillators with positive energies

$$\zeta_\beta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{l=-\infty}^\infty \exp\left\{-\left(\frac{2\pi l}{\beta}\right)^2 t\right\} \Theta(t) dt, \quad (\text{A.52})$$

where

$$\Theta(t) = \sum_{\omega} \exp\{-\omega^2 t\}, \quad \omega^2 > 0. \quad (\text{A.53})$$

Transforming the sum over  $l$  in (A.52) with the use of the theta function identity (A.47) and using the integral representation for the Kelvin functions

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty dt \, t^{-\nu-1} \exp\left\{-t - \frac{z^2}{4t}\right\}, \quad (\text{A.54})$$

one obtains [35]

$$\zeta_\beta(s) = \frac{\beta}{2\sqrt{\pi}} \frac{1}{\Gamma(s)} Y_{\text{spat}}\left(s - \frac{1}{2}\right) + \frac{2\beta}{\sqrt{\pi}\Gamma(s)} \sum_{l=1}^\infty \sum_{\omega} \left(\frac{\beta l}{2\omega}\right)^{s-1/2} K_{1/2-s}(\beta l \omega), \quad (\text{A.55})$$

where

$$Y_{\text{spat}}(s) = \int_0^\infty dt \, t^{s-1} \Theta(t) = \zeta_{\text{spat}}(s) \Gamma(s). \quad (\text{A.56})$$

This function has the following pole structure [33]

$$Y_{\text{spat}}(s) = \frac{1}{(4\pi)^{3/2}} \sum_r \frac{C_r}{s+r-3/2} + f(s), \quad (\text{A.57})$$

where  $C_r$  are defined by the asymptotic expansion of  $\Theta(t)$ , and  $f(r)$  is an entire analytic function of  $s$ . This relation implies

$$Y_{\text{spat}}(s - \frac{1}{2}) = \frac{C_2}{(4\pi)^{3/2}} \frac{1}{s} + \text{PPY}_{\text{spat}}(-\frac{1}{2}). \quad (\text{A.58})$$

Taking Eq.(A.56) and the properties of the gamma function into account,

$$\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3), \quad \Gamma(-\frac{1}{2} + s) = -2\sqrt{\pi} \{1 + (-\gamma + 2 - 2 \ln 2)s\} + O(s^3), \quad (\text{A.59})$$

one therefore obtains

$$\frac{1}{2\sqrt{\pi}\Gamma(s)} Y_{\text{spat}}(s - \frac{1}{2}) = \frac{C_2}{16\pi^2} + \left\{ \frac{(1 - \ln 2)C_2}{8\pi^2} - \text{PP}\zeta_{\text{spat}}(-\frac{1}{2}) \right\} s + O(s^2). \quad (\text{A.60})$$

Finally, using

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad \sum_{l=1}^{\infty} \frac{1}{l} e^{-lz} = -\ln(1 - e^{-z}), \quad (\text{A.61})$$

one arrives at

$$\zeta_{\beta}(s) = \frac{C_2\beta}{16\pi^2} + \left\{ \left( \frac{(1 - \ln 2)C_2}{8\pi^2} - \text{PP}\zeta_{\text{spat}}(-\frac{1}{2}) \right) \beta - 2 \sum_{\omega} \ln(1 - e^{-\beta\omega}) \right\} s + O(s^2), \quad (\text{A.62})$$

which gives rise to the formula (6.19) in the main text.

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